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AXISYMMETRIC CONTACT PROBLEM TAKING ACCOUNT OF ADHESION AND SLIP

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There is considered the problem of impression of a stamp in the shape of a body of revolution in a transversely-isotropic half-space, with friction and adhesion taken into account. A method based on expanding the solution in a series in a parameter dependent on the ratio of the stiffness characteristics is used. The case of a stamp with a flat base is considered as an illustration. Approximate relationships are obtained to determine the radius of the adhesion area and the contact stress distribution.

Let a stamp in the shape of a body of revolution be impressed in a transversely-isotropic half-space. The contact domain consists of the friction part abutting on the boundary of the contact domain, and the adhesion part. Because of symmetry the contact domain and the adhesion part will be concentric circles with a common center on the stamp axis. The radius of the circle separating the friction and adhesion parts is not known beforehand and should be determined during solution of the problem. It is required to determine the normal and tangential forces in the contact domain also.

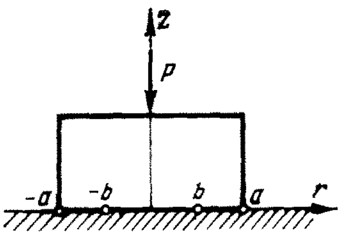

Fig. 1

The problem reduces to solving the equilibrium equations for a transversely-isotropic medium

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{k}\frac{\partial^2 w}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)=0 \tag{1.1}$$

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+k\varepsilon\frac{\partial^2 u}{\partial z^2}+k\varepsilon\frac{\partial^2 w}{\partial r\,\partial z}=0,\quad k=\frac{G'}{E'},\quad \varepsilon=\frac{E'}{E}$$

under the following boundary conditions outside the contact domain $(r>a)$, in the whole contact domain $(r<a)$, on the adhesion part $(0<r<b)$ and on the slip part $(b<r<a)$:

$$\sigma_z=\tau_{rz}=0,\quad r>a;\quad w=-C+f(r),\quad r<a \tag{1.2}$$

$$u=0,\quad 0<r<b;\quad \tau_{rz}=-\rho\sigma_z,\quad b<r<a$$

In addition

$$w\to 0,\quad u\to 0\quad \text{when}\quad \sqrt{r^2+z^2}\to\infty$$

Here a is the radius of the contact domain, b is the radius of the adhesion section, $f(r)$ is a function describing the shape of the stamp, E and E' are the tension-compression elastic moduli in the plane of isotropy and in a normal direction, G' is the shear modulus in a plane normal to the plane of isotropy, w and u are displacement vector components in the directions of the z and r axes, respectively, and ρ is the friction coefficient. The Poisson's ratio are taken equal to zero.

Let us introduce the transformation of variables

$$r = r_1, \quad z = k^{-1/2}z_1, \quad w = W^1, \quad u = \varepsilon U^1 \tag{1.3}$$

$$r = r_2, \quad z = \varepsilon^{1/2}k^{1/2}z_2, \quad w = \varepsilon W^2, \quad u = \varepsilon^{1/2}U^2 \tag{1.4}$$

Substituting (1.3) and (1.4) into (1.1), we obtain, respectively

$$W_{rs}^1 + W_{zz}^1 + \varepsilon k^{1/2}U_{sz}^1 = 0, \quad U_{sr}^1 + \varepsilon k^2 U_{zz}^1 + k^{3/2}W_{rz}^1 = 0 \tag{1.5}$$

$$\varepsilon k^2 W_{rs}^2 + W_{zz}^2 + k^{3/2}U_{sz}^2 = 0, \quad U_{sr}^2 + U_{zz}^2 + \varepsilon k^{1/2}W_{rz}^2 = 0 \tag{1.6}$$

Here

$$\varphi_s^{\,l} = \frac{\partial \varphi^l}{\partial r_l} + \frac{\varphi^l}{r_l}, \quad \varphi_{sz}^{\,l} = \frac{\partial}{\partial z_l}\left(\frac{\partial \varphi^l}{\partial r_l} + \frac{\varphi^l}{r_l}\right)$$

$$\varphi_{sr}^{\,l} = \frac{\partial}{\partial r_l}(\varphi_s^{\,l}), \quad \varphi_{rs}^{\,l} = \left(\frac{\partial \varphi^l}{\partial r_l}\right)_s = \frac{\partial^2 \varphi^l}{\partial r_l^2} + \frac{1}{r_l}\frac{\partial \varphi^l}{\partial r_l} \quad (l = 1, 2)$$

Let us assume that $E > E' \sim G'$; then ε can be considered a small parameter in an asymptotic analysis of (1.5) and (1.6). The solution obtained by asymptotic integration of the system of the first kind (1.5) corresponds to a stress-strain state varying relatively slowly along the z axis as compared with the corresponding solution of the system of the second kind (1.6), which is of a boundary layer nature [1].

Let us represent the displacement vector components in the form of the sum of solutions of both kinds

$$u = u_1 + u_2, \quad w = w_1 + w_2 \tag{1.7}$$

We seek the functions W^l and U^l $(l = 1, 2)$ in the form of asymptotic series in the parameter $\varepsilon^{1/2}$

$$W^l = \sum_{n=0}^{\infty} \varepsilon^{n/2} W^{l,n} = \sum_{i=0}^{\infty}\sum_{j=0}^{1} \varepsilon^{i+j/2} W^{l,2i+j} \tag{1.8}$$

$$U^l = \sum_{n=0}^{\infty} \varepsilon^{n/2} U^{l,n} = \sum_{i=0}^{\infty}\sum_{j=0}^{1} \varepsilon^{i+j/2} U^{l,2i+j}$$

An additional coordinate transformation is hence introduced

$$\zeta_1 = z_1 \sum_{i=0}^{\infty} \varepsilon^i \alpha_i, \quad \zeta_2 = z_2 \sum_{i=0}^{\infty} \varepsilon^i \beta_i \tag{1.9}$$

with the undetermined coefficients α_i and β_i $(i = 0, 1, \ldots)$.

Substituting (1.8) and (1.9) into (1.5) and (1.6) and the boundary conditions, and splitting the expressions obtained in powers of $\varepsilon^{1/2}$, we obtain the following equations and boundary conditions for the functions $W^{l,2i+j}$ and $U^{l,2i+j}$.

The stress-strain state of the first kind is

$$W_{rs}^{1,2i+j} + W_{\zeta\zeta}^{1,2i+j} b_0 = -\sum_{\nu=0}^{i-1} (W_{\zeta\zeta}^{1,2\nu+j} b_{i-\nu} + k^{1/2} U_{s\zeta}^{1,2\nu+j} \alpha_{i-1-\nu}) \tag{1.10}$$

$$U_{sr}^{1,2i+j} = -k^2 \sum_{\nu=0}^{i-1} U_{\zeta\zeta}^{1,2\nu+j} b_{i-1-\nu} - k^{3/2} \sum_{\nu=0}^{i} W_{r\zeta}^{1,2\nu+j} \alpha_{i-\nu}$$

$$b_p = \sum_{i=0}^{p} \alpha_i \alpha_{p-i} \quad (i = 0, 1, \ldots)$$

The boundary conditions are:
for $i = 0,\ j = 0$

$$W^{1,0} = -C + f(r), \quad r < a; \quad W_\zeta^{1,0} = 0, \quad r > a \tag{1.11}$$

for all the remaining $i,\ j$

$$W^{1,2i+j} = -W^{2,2(i-1)+j}, \quad r < a \tag{1.12}$$

$$W_\zeta^{1,2i+j} \alpha_0 = -\sum_{\nu=0}^{i-1} W_\zeta^{1,2\nu+j} \alpha_{i-\nu} - k^{-1} \sum_{\nu=0}^{i} W_\zeta^{2,2\nu+j-1} \beta_{i-\nu}, \quad r > a$$

The stress-strain state of the second kind is

$$W_{\zeta\zeta}^{2,2i+j} c_0 = -\sum_{\nu=0}^{i-1} W_{\zeta\zeta}^{2,2\nu+j} c_{i-\nu} - k^{3/2} \sum_{\nu=0}^{i} U_{s\zeta}^{2,2\nu+j} \beta_{i-\nu} - k^2 W_{rs}^{2,2(i-1)+j} \tag{1.13}$$

$$U_{sr}^{2,2i+j} + U_{\zeta\zeta}^{2,2i+j} c_0 = -\sum_{\nu=0}^{i-1} (U_{\zeta\zeta}^{2,2\nu+j} c_{i-\nu} - k^{1/2} W_{r\zeta}^{2,2\nu+j} \beta_{i-1-\nu})$$

$$c_p = \sum_{i=0}^{p} \beta_i \beta_{p-i}$$

The boundary conditions are

$$U^{2,2i+j} = -U^{1,2i+j-1}, \quad r < b \tag{1.14}$$

$$U_\zeta^{2,2i+j} \beta_0 = -\sum_{\nu=0}^{i-1} (U_\zeta^{2,2\nu+j} \beta_{i-\nu} + k U_\zeta^{1,2\nu+j} \alpha_{i-1-\nu}) -$$

$$k^{1/2} (W_r^{1,2i+j} + W_r^{2,2(i-1)+j}) - \rho \sum_{\nu=0}^{i} (W_\zeta^{1,2\nu+j} \alpha_{i-\nu} + k^{-1} W_\zeta^{2,2i+j-1} \beta_{i-\nu})$$

$$r > b$$

It should be taken into account that if any upper limit of summation in (1.10) and (1.12)–(1.14) is negative, then this sum equals zero. Analogously, if any function of the second superscript (denoting the number of the approximation) is negative, then this function is zero. For instance, for $i = 0$ we obtain from (1.10) and (1.13)

$$W_{rs}^{1,j} + W_{\zeta\zeta}^{1,j} b_0 = 0, \quad U_{sr}^{1,j} = -k^{3/2} W_{r\zeta}^{1,j}$$

$$W_{\zeta\zeta}^{2,j} c_0 = -k^{3/2} U_{s\zeta}^{2,j} \beta_0, \quad U_{sr}^{2,j} + U_{\zeta\zeta}^{2,j} c_0 = 0$$

The first equations (for the functions $W^{1,2i+j}$) in the system (1.10) as well as the second equations (for the functions $U^{2,2i+j}$) in the system (1.13) will be called basic equations.

The following theorem is valid for the basic equations: if the coefficients α_i and β_i are determined by the formulas

$$\alpha_0 = 1, \quad 2\alpha_{p+1} = k^2\gamma_p + \sum_{m=1}^{p} (k^2\gamma_{p-m} - \alpha_{p+1-m})\,\alpha_m \tag{1.15}$$
$$\beta_0 = 1, \quad 2\beta_{p+1} = -k^2\delta_p - \sum_{m=1}^{p} (k^2\delta_{p-m} + \beta_{p+1-m})\,\beta_m$$
$$\gamma_0 = \delta_0 = 1, \quad \gamma_n = \alpha_n + k^2 \sum_{j=0}^{n-1} b_j\gamma_{n-1-j}$$
$$\delta_n = \beta_n + k^2\delta_{n-1} - \sum_{j=0}^{n-1} \delta_j c_{n-j}$$

then the fundamental equations have the form

$$W_{rs}^{1,2i+j} + W_{\zeta\zeta}^{1,2i+j} = 0, \quad U_{sr}^{2,2i+j} + U_{\zeta\zeta}^{2,2i+j} = 0 \tag{1.16}$$
$$(i = 0, 1, \ldots; j = 0, 1)$$

The proof of this theorem is presented in Sect. 3.

We shall henceforth consider the coefficients α_i, β_i to be defined by (1.15).

It is seen from the expansions presented that the boundary conditions for the functions $W^{1,2i+j}$ are satisfied in the solution of the first equations of the system (1.10) describing the stress-strain state of the first kind. The functions $U^{1,2i+j}$ are defined as particular solutions of the second equations of this system.

The boundary conditions for the system (1.13), describing the stress-strain state of the second kind, are determined after the appropriate equations of the first kind have been solved. These boundary conditions are satisfied in the solution of the second equations of the system (1.13). The functions $W^{2,2i+j}$ are found as particular solutions of the first equations of this system.

The problem therefore reduces to successive integration of (1.16) for the functions $W^{1,2i+j}$ and the functions $U^{2,2i+j}$. Finding the functions $U^{1,2i+j}$ and $W^{2,2i+j}$ is not difficult.

The exact solution of (1.16) can be obtained by using integral transformations [2]. It can be shown that the functions $W^{1,2i+j}$ and $U^{2,2i+j}$ are continuous in the whole domain of definition, and their derivatives are continuous everywhere except at the points $r = b$, $z = 0$ and $r = a$, $z = 0$, where they have integrable singularities. It hence follows that the solution is asymptotic in nature everywhere with the exception of arbitrarily small neighborhoods of the above-mentioned points.

After the functions $W^{l,2i+j}$ and $U^{l,2i+j}$ $(l = 1, 2;\ i = 0, 1, \ldots;\ j = 0, 1)$ have been determined, the stresses σ_z and τ_{rz} are found from the formulas

$$\sigma_z = E' \sum_{i=0}^{\infty} \sum_{j=0}^{1} \varepsilon^{i+j/2} \sum_{\nu=0}^{i} (k^{1/2} W_\zeta^{1,2\nu+j}\alpha_{i-\nu} + k^{-1/2}\varepsilon^{1/2} W_\zeta^{2,2\nu+j}\beta_{i-\nu})$$
$$\tau_{rz} = G' \sum_{i=0}^{\infty} \sum_{j=0}^{1} \varepsilon^{i+j/2} \Big[\sum_{\nu=0}^{i} (k^{1/2}\varepsilon U_\zeta^{1,2\nu+j}\alpha_{i-\nu} + k^{-1/2} U_\zeta^{2,2\nu+j}\beta_{i-\nu}) +$$

$$W_r^{1,\,2i+j} + \varepsilon W_r^{2,\,2i+j}\Big]$$

The unknown boundary of the adhesion and friction parts is found from the condition of continuity of the tangential stresses on this boundary.

The constant C in (1.11) (the settlement of the stamp) is determined from the stamp equilibrium condition (P is the magnitude of the clamping force)

$$P + 2\pi \int_0^a \sigma_z r dr = 0 \tag{1.17}$$

2. As an illustration, let us solve the problem for a stamp with a flat base ($f(r) \equiv 0$). The solution (taking account of one approximation) reduces to integrating (2.1) with the boundary conditions (2.2)

$$W_{rs}^{1,0} + W_{\zeta\zeta}^{1,0} = 0, \quad U_{sr}^{2,0} + U_{\zeta\zeta}^{2,0} = 0 \tag{2.1}$$

$$\zeta_1 = 0, \quad W^{1,0} = -C, \quad r < a; \quad W_\zeta^{1,0} = 0, \quad r > a \tag{2.2}$$

$$\zeta_2 = 0, \quad U^{2,0} = 0, \quad r < b; \quad U_\zeta^{2,0} = \begin{cases} -\rho W_\zeta^{1,0}, & b < r < a \\ -k^{1/2} W_r^{1,0}, & r > a \end{cases}$$

The function $W^{1,0}$ has the form (the constant C is determined from the condition (1.17))

$$W^{1,0} = -2C\pi^{-1} \arcsin \left[2a \left(\sqrt{\zeta_1^2 + (r+a)^2} + \sqrt{\zeta_1^2 + (a-r)^2}\right)^{-1}\right] \tag{2.3}$$

$$C = Pk^{-1/2} / (4aE')$$

We seek the solution of (2.1) in the form ($J_1(x)$ is the first order Bessel function)

$$U^{2,0} = \int_0^\infty A(p) \exp(p\zeta_2) J_1(pr)\, dp$$

Substituting this expression into the second pair of boundary conditions (2.2) written with (2.3) taken into account, we obtain a system of dual integral equations to determine the function $A(p)$

$$\int_0^\infty A(p) J_1(pr)\, dp = 0, \quad r > b$$

$$\int_0^\infty A(p) J_1(pr)\, p\, dp = \begin{cases} 2C\rho\pi^{-1}(a^2 - r^2)^{-1/2}, & b < r < a \\ -2Cak^{-1/2}(\pi r)^{-1}(r^2 - a^2)^{-1/2}, & r > a \end{cases}$$

The solution of these equations has the form ($\mathrm{K}(x)$ is the complete elliptic integral of the first kind) [2]

$$A(p) = \frac{2C\rho}{\pi^2}\left[-k^{1/2}\pi \sin(ap) + B(p)\right]$$

$$B(p) = \frac{2\rho}{a} p \int_b^a \mathrm{K}'\left(\frac{x}{a}\right) x \sin(px)\, dx + pk^{1/2} \int_0^b \ln\left(\frac{a+x}{a-x}\right) \sin(px)\, dx$$

The normal contact stresses and the tangential stresses under a flat stamp are determined by the formulas

$$\sigma_z = -(2\pi a)^{-1} P (a^2 - r^2)^{-1/2}, \quad \tau_{rz} = G' k^{-1/2} U_{\zeta_2}^{2,0}$$

It is clear from physical considerations that the tangential stresses should be continuous on the interface of the adhesion and slip zones, therefore, the derivative $U_\zeta^{2,0}$ should be continuous on this boundary. We have

$$\zeta_2 = 0, \quad U_\zeta^{2,0} = 2C\rho\pi (a^2 - r^2)^{-1/2}, \quad b \leqslant r \leqslant a \tag{2.4}$$

$$U_\zeta^{2,0} = -\frac{2Ck^{1/2}}{\pi^2} \int_0^\infty \sin(ap) J_1(rp)\, dp + \frac{2C}{\pi} \int_0^\infty B(p) J_1(pr)\, dp, \quad 0 \leqslant r < b$$

Integrating the inner integral by parts twice, we reduce the last formula to

$$\zeta_2 = 0, \quad U_\zeta^{2,0} = 2C\pi^{-1} (G_1 + G_2 + I_1 + I_2)$$

$$G_1 = \frac{\rho\pi r}{a\sqrt{a^2 - r^2}} + r\left[\frac{ak^{1/2}}{a^2 - b^2} - \frac{2\rho}{a}\frac{d}{ab}\left(b\mathbf{K}'\left(\frac{b}{a}\right)\right)\right](b + (b^2 - r^2)^{1/2})^{-1}$$

$$G_2 = \left[k^{1/2} \ln \frac{a+b}{a-b} - \frac{2\rho b}{a} \mathbf{K}'\left(\frac{b}{a}\right)\right] r (b^2 - r^2)^{-1/2} (b + (b^2 - r^2)^{1/2})^{-1}$$

$$I_1 = \int_0^N f_1(pr)\, dp, \quad I_2 = \int_N^\infty f_1(pr)\, dp \quad (N > 0)$$

$$f_1(pr) = 2\left[\frac{\rho}{a}\int_b^a \frac{d^2}{dx^2}\left(x\mathbf{K}'\left(\frac{x}{a}\right)\right) \sin(px)\, dx + ak^{1/2}\int_0^b \frac{x\sin(px)}{(a^2 - x^2)^2}\, dx\right]\frac{J_1(pr)}{p}$$

The function G_1 is continuous in r in the interval $0 \leqslant r \leqslant b < a$. The integral I_1 is also continuous in this interval (as a definite integral of the continuous function f_1). The improper integral I_2 converges uniformly (for large p the function f_1 admists the estimate $f_1 < B / p^2$ [3]), and therefore converges to a continuous function. The function G_2 undergoes a discontinuity at $r = b$. Hence, $G_2 \equiv 0$ is necessary for the continuity of the derivative $U_\zeta^{2,0}$. Consequently

$$k^{1/2} \ln ((1 + b / a) / (1 - b / a)) = 2\rho (b / a) \mathbf{K}' (b / a) \tag{2.5}$$

The relationship (2.5) determines the boundary of the adhesion and friction parts not known earlier.

After simple, but awkward manipulations, we obtain the following formula for the tangential stresses on the adhesion part ($\Pi_1(n, x)$ is the complete elliptic integral of the third kind):

$$\tau_{rz} = {}^1/_2 P k^{1/2} \pi^{-2} a^{-2} t^{-1} (1 - t^2)^{-1/2} \{\rho\pi k^{-1/2} t + (\beta_*^2 - t^2)^{1/2} [\beta_*^{-1} \ln ((1 + \beta_*) / (1 - \beta_*)) - 2\rho k^{-1/2} t^2 \Pi_1 (t^2 - 1, \beta_*) / (1 - t^2)] (1 - t^2)^{-1/2} - \ln ((1 - t^2)^{1/2} + (\beta_*^2 - t^2)^{1/2}) / ((1 - t^2)^{1/2} - (\beta_*^2 - t^2)^{1/2}))\}, \quad t = r / a, \ \beta_* = b / a \quad (0 \leqslant t \leqslant \beta_*)$$

On the slip part

$$\tau_{rz} = {}^1/_2 P\rho\pi^{-1} a^{-2} (1 - t^2)^{-1/2} \quad (\beta_* \leqslant t < 1)$$

The dependence of the quantity b / a (the ratio of the adhesion part radius to the stamp radius) on the friction coefficient ρ for $k = 1/3$ is shown in Fig. 2 (curve *1*) and the distribution of the dimensionless tangential stresses $T_1 = \tau_{rz} 2\pi a / P$ in the contact domain is presented in Fig. 3 for $k = 1/3$, $\rho = 0.3$. The point $\beta_* = b / a$ separates the adhesion and friction parts.

It should be noted that in the solution obtained (taking just one approximation into account) the singularity in the contact stresses on the boundary of the contact domain has the form $(a - r)^{-1/2}$ while the exact solution of the problem in the presence of Coulomb friction should contain the singularity $(a - r)^{-1/2+\theta(\rho,\varepsilon)}$ exactly as for the plane problem (this follows from the fact that the equations of three-dimensional elasticity theory reduce to the plane problem and complex shear in the neighborhood of the singular line [4]). Therefore, the epxression for θ is known [5]. The series expansion in ε has the form

$$(a - r)^{-1/2+\theta} = (a - r)^{-1/2} \{1 + \varepsilon^{1/2}\rho k^{1/2} \ln (a - r) + \varepsilon [\rho k^{3/2} \ln (a - r) + {}^1/_2\rho^2 k \ln^2 (a - r)] + \ldots\} \tag{2.6}$$

The singularity obtained in this paper agrees with the first term of the expansion. Subsequent approximations will evidently contain appropriate corrections.

The relative error of any partial sum of the series (2.6) becomes arbitrarily large as $r \to a$. Meanwhile, uniform accuracy in the whole contact domain can be achieved by "matching" the approximate and "singular" solutions, which has the form

$$\sigma_z^* = A (a - r)^{-1/2+\theta}$$

The constant factor A is determined from the matching conditions which are given as follows: both the approximate and singular solutions and their derivatives with respect to r should agree in a certain neighborhood of $r = r_0$ i.e., for $r = r_0$, $z = 0$

$$\sigma_z = \sigma_z^*, \quad \partial\sigma_z / \partial r = \partial\sigma_z^* / \partial r \tag{2.7}$$

The conditions (2.7) permit the determination of the radius r_0 , and the constants of the approximate and singular solutions, in combination with the integral equilibrium condition of the stamp. The dependence of the position of the matching line on the coefficient ρ is shown in Fig. 2 for $k = 1/3$ and $\varepsilon = 1$ (curve *2*) and $\varepsilon = 1/3$ (curve *3*). To obtain the solution which is uniformly suitable in the whole domain, the singular solution must be used for $r_0 \leqslant r \leqslant a$. Incidence of the matching points in the adhesion zone $(r_0 < b)$ indicates the need to take account of higher

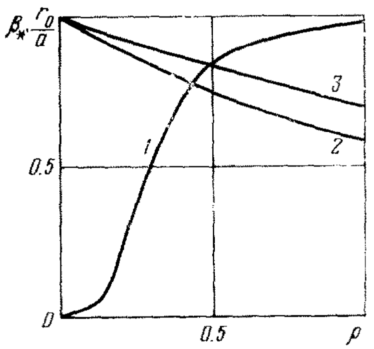

Fig. 2

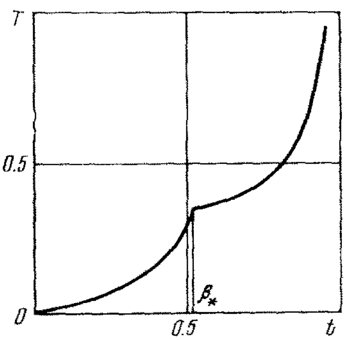

Fig. 3

approximations. It follows from Fig. 2 that this holds for large ρ ($\rho > 0.4$), and for real friction coefficients ($\rho < 0.3$), the zones in which the use of the singular solution is necessary are small and occupy less than 20% (on the radius) of the contact domain even when taking just one approximation into account.

3. Let us show that if the coefficients α_i and β_i are determined from (1.15), then the fundamental equations have the form (1.16). We prove this for the system (1.10). The proof is by induction.

For $i = 0$ we have

$$W_{rs}^{1,j} + W_{\zeta\zeta}^{1,j} b_0 = 0, \quad U_{sr}^{1,j} = -k^{3/2} W_{\zeta r}^{1,j} \alpha_0 \tag{3.1}$$

In order for the first equation of the system (3.1) to have the form (1.16), it is sufficient to set $\alpha_0 = 1$ (hence $b_0 = 1$).

For $i = 1$

$$W_{rs}^{1,2+j} + W_{\zeta\zeta}^{1,2+j} = -W_{\zeta\zeta}^{1,j} b_1 - k^{1/2} U_{s\zeta}^{1,j} \alpha_0 \tag{3.2}$$

$$U_{sr}^{1,2+j} = -k^2 U_{\zeta\zeta}^{1,j} - k^{3/2} W_{rs}^{1,j} \alpha_1 - k^{3/2} W_{r\zeta}^{1,2+j} \alpha_0$$

Let us integrate the second equation of (3.1) with respect to r and let us differentiate with respect to ζ. We obtain

$$U_{s\zeta}^{1,j} = -k^{3/2} W_{\zeta\zeta}^{1,j} \alpha_0 \tag{3.3}$$

Taking account of the condition at infinity, we set the arbitrary function which appears during the integration equal to zero. After substituting (3.3) into the first equation of (3.2), we find

$$W_{rs}^{1,2+j} + W_{\zeta\zeta}^{1,2+j} = W_{\zeta\zeta}^{1,j} (b_1 - k^2 \alpha_0^2)$$

Equating coefficients of $W_{\zeta\zeta}^{1,j}$ to zero (and taking into account that $\alpha_0 = 1$ and $b_1 = 2\alpha_1$), we obtain $\alpha_1 = k^2 / 2$.

Therefore, the theorem is valid for $i = 0,1$. Let us assume the theorem to be valid for $i \leqslant p$, and let us prove it for $i = p + 1$.

We show that if the functions $W^{1,2i+j}$ $(i = 0, 1, \ldots, p)$ satisfy the first equation in (1.16), then the functions $U^{1,2i+j}$ $(i = 0, 1, \ldots, p)$, determined from the second equations of the system (1.10) will satisfy the equation

$$U_{sr}^{1,2i+j} + U_{\zeta\zeta}^{1,2i+j} = 0 \tag{3.4}$$

If the functions $U^{1,2i+j}$ are found from the equation

$$U_{sr}^{1,2i+j} = f_2$$

where f_2 satisfies (3.4), then $U^{1,2i+j}$ is the solution of this equation. Indeed

$$U^{1,2i+j} = r^{-1} \int \left(r \int f_2 \, dr \right) dr$$

$$U_{sr}^{1,2i+j} + U_{\zeta\zeta}^{1,2i+j} = r^{-1} \int \left(r \int (f_{2sr} + f_{2\zeta\zeta}) \, dr \right) dr = 0$$

It therefore remains to show that the right sides of the second equations of the system (1.10) satisfy (3.4) for $i \leqslant p$.

It is seen that if the function $W^{1,2i+j}$ satisfies (1.16), then its derivative $W_r^{1,2i+j}$ satisfies (3.4). The right side of the second equation of (1.10) now evidently satisfies (3.4) for $i=0$, and this is proved by induction for $i=1,2,\ldots,p$.

In conformity with (3.4), we substitute the quantity $U_{rs}^{1,2i+j}$ in the second equations of (1.10) in place of $U_{\zeta\zeta}^{1,2i+j}$, we integrate the relationships obtained with respect to r and differentiate with respect to ζ. We obtain

$$U_{s\zeta}^{1,\,2i+j} = k^2 \sum_{\nu=0}^{i-1} U_{s\zeta}^{1,\,2\nu+j} b_{i-1-\nu} - k^{3/2} \sum_{\nu=0}^{i} W_{\zeta\zeta}^{1,\,2\nu+j} a_{i-\nu} \quad (i=0,1,\ldots,p) \qquad (3.5)$$

The system (3.5) can be considered as a system of linear algebraic equations in $U_{s\zeta}^{1,2i+j}$. This system has a triangular matrix of coefficients with nonzero elements along the principal diagonal, and therefore, is solvable everywhere. The solution of the system (3.5) has the form

$$U_{s\zeta}^{1,\,2i+j} = -k^{3/2} \sum_{\nu=0}^{i} W_{\zeta\zeta}^{1,\,2\nu+j} \gamma_{i-\nu} \quad (i=0,1,\ldots,p) \qquad (3.6)$$

where the quantities γ_n $(n=0,1,\ldots)$ are determined by the recursion formulas in (1.15).

Substituting (3.6) into the $(p+1)$-th fundamental equation of the system (1.10), we obtain

$$W_{rs}^{1,\,2(p+1)+j} + W_{\zeta\zeta}^{1,\,2(p+1)+j} = -\sum_{\nu=0}^{p} W_{\zeta\zeta}^{1,\,2\nu+j} \Big(b_{p+1-\nu} - k^2 \sum_{m=0}^{p-\nu} \alpha_m \gamma_{p-\nu-m} \Big) \qquad (3.7)$$

Because of the selection of the coefficients α_i mentioned in (1.15), all the coefficients in the right side of (3.7) vanish. Indeed, from the relationship for $\alpha_{p+1-\nu}$, it follows according to (1.15)

$$\sum_{m=0}^{p+1-\nu} \alpha_m \alpha_{p+1-\nu-m} - \sum_{m=0}^{p-\nu} k^2 \gamma_{p-\nu-m} \alpha_m = 0$$

i.e., the expression in the parentheses in (3.7) vanishes.

Therefore, the basic equations of the system (1.10) will have the form of the first relationship of (1.16).

The proof for the system (1.13) is analogous.

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